Determining the viscosity of water using the Hagen-Poisseuille relation

A. Renn (and L. Richardson-Foulger, D. Quirk, P. Kerner)
L2 Laboratory Skills and Electronics, Lab Group PH228, Tuesday

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This report uses the Hagen-Poiseuille relation to determine the viscosity of water using a method involving water flow from a reservoir through a capillary tube. The final viscosity determined is 0.93 ± 0.05 mPa s, which is two standard deviations from the literature value. This proves the method a valid method of viscometry.

1. INTRODUCTION

It is useful in physics to have a robust method of measuring material properties in order to use them to further develop theories, in this case viscosity is useful in medicine or oil drilling for example. In this report we will be asking the question ‘is the Hagen-Poiseuille relation a good way to determine the viscosity of water?’.

In 1838, Poiseuille discovered that for a liquid of viscosity η in laminar flow, the flow rate through a tube of radius a and length L, where $L \gg a$, is given by,

$$\frac{dV}{dt} = \frac{\pi}{8}\frac{\rho g h}{\eta}\frac{a^4}{L}, \tag{1}$$

where $\frac{dV}{dt}$ is the volume of the water V passing a point per unit time t, and $\rho g h$ is the pressure difference at two ends of a tube, ρ being the liquid’s density, g local gravity, and h the height of liquid from one end of the tube (this is Bernoulli’s relation for pressure as a function of depth[2]).

By use of an experiment which allows the altering of h and distinct measurements of $\frac{dV}{dt}$ for each h, one can plot these variables and determine a relation between the liquid viscosity η and gradient m as follows:

$$\eta = \frac{\pi}{8}\frac{\rho g h}{m}\frac{a^4}{L}. \tag{2}$$

In this report we will use literature values of 9.81 m s^{-2} for g, and 997 kg m^{-3} for ρ[2]. We will use the following equation for the literature value for η for water, measuring a lab temperature of T,

$$\eta = A\exp\left(\frac{B}{T} + CT + DT^2\right), \tag{3}$$

where $A = 1.856 \times 10^{-11}$ mPa s, $B = 4209$ K, $C = 0.04527$ K^{-1}, and $D =$ -3.376 $\times 10^{-5}$ K^{-2} [3].

2. METHODS

As seen in Figure 1, a capillary tube was attached to the bottom of a reservoir of tap-water at a consistent temperature, which was in turn suspended above a beaker on a mass balance. The data logger attached to the mass balance recorded the mass of water measured over a time period of 300 2and 120 s for the white and blue tube respectively. This was chosen as in this range, the height of water in the water bath stayed relatively constant (change of x in time t), which is required for our assumption of a linear relation between height and flow rate. The flow rate was then calculated using chi-squared analysis for each height, and then the gradient of the graph of flow rate against height was determined again by chi squared analysis to determine a value of m (see Introduction) with which the viscosity of water could be determined.

The length of each capillary tube was measured with a ruler, whereas the diameter of each tube was measured at 8 different angles to the observer using a travelling microscope and a weighted mean of these measurements was calculated. This was necessary because the radius has a fourth power in the Hagen-Poisseuille relation, meaning the error in the radius has a large impact on the error on the final result.

The height of the surface of the water from the bottom of the reservoir was determined using a ruler submerged in the water bath as close to the capillary tube hole as possible, as to minimise parallax error. Then, the height of the capillary tube from the bottom of the water was measured and subtracted from the height of the surface to determine the height of the surface from the capillary tube.

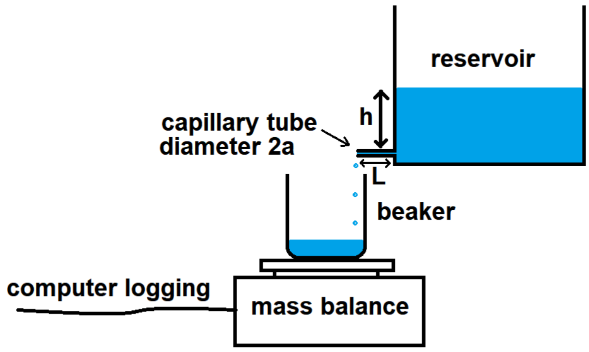

FIG. 1: A diagram of the experimental setup.

3. RESULTS

Figure 4 was created by determining the gradients of each of the lines on figures 2 and 3, using χ^2-minimisation, and plotting them as a function of height.

From figure 4, a gradient for each line was found using χ^2-minimisation. Equation (2) was used to determine a value for water viscosity, using also the measured diameters, both of which are values can be found in table 1. The values and associated errors were deemed similar enough to be combined into a final value for the viscosity of water of 0.93 ± 0.05 mPa s.

The measured temperature of the water was 18.4 ± 0.1 ^{o}C, which, using equation (3), results in a literature value of $\eta = 1.040 \pm 0.003$ mPa s. This is also shown in table 1.

Capillary tube	Diameter (mm)	Measured viscosity (mPa s)
White	0.58 ± 0.01	0.93 ± 0.06
Blue	0.359 ± 0.007	1.00 ± 0.08
Combined	-	0.97 ± 0.05
Lab value	-	1.057 ± 0.003

TABLE I: Values of viscosity and their associated uncertainties obtained from the tubes listed. The methods used to calculate the uncertainties are described in Appendix I.

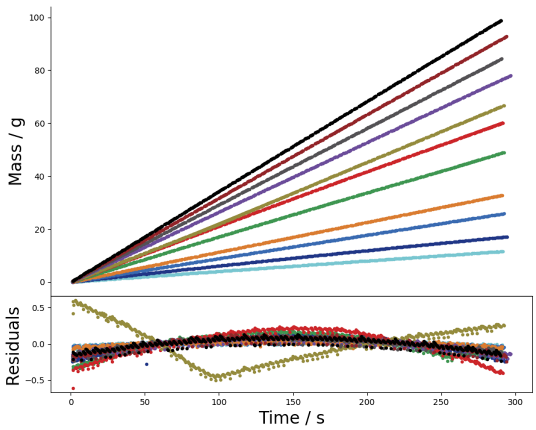

FIG. 2: A graph of mass against time for the system using the white capillary tube. Note the oddly shaped residual plot.

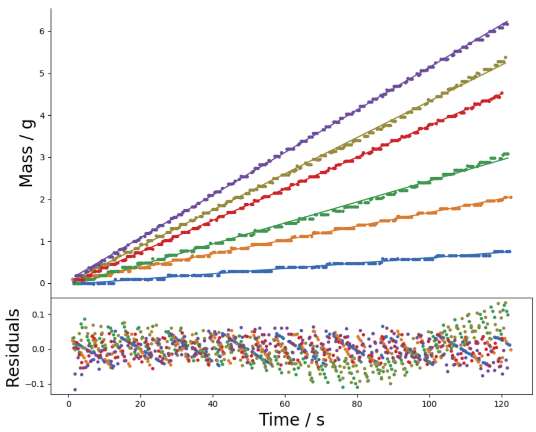

FIG. 3: A graph of mass against time for the system using the blue capillary tube. Note the periodic residual plot.

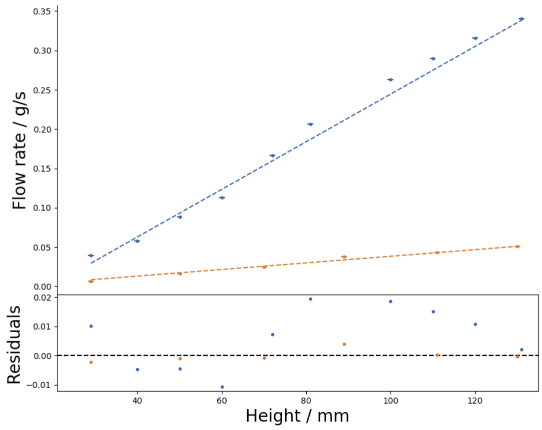

FIG. 4: A comparison of flow rate vs height for both tubes, from which an estimate of the gradient, and thus viscosity, can be made.

4. DISCUSSION

The 89 cm curve in figure 2 shows a residual shape which differs from all other heights. This is presumed to be because the experiment was tampered with while data-collection was happening. Therefore, this gradient was removed from figure 4.

The other residuals in figure 2 show a quadratic trend. This may be because height was assumed to be constant, but the time period over which measurements were made meant that substantial water drained out. It was calculated that over 1 minute, 0.4 mm was lost from the height of the water using the white tube, so over the time of 5 minutes in figure 2, 2 mm will have been lost. It was assumed this would be negligible, but clearly this was not the case. One can see that for higher flow rates, this has more of an effect as the residual trend is a steeper curve. This non-linearity of the curves may have lead to the value of η for the white tube to be wrongly estimated. However the same cannot be observed for the blue tube as less water escaped due to the smaller diameter, and because it was over a shorter timescale of 2 minutes. Thus, the value of η from the blue tube may be more accurate.

The linear fit in figure 4 looks to be adequate, but the vertical errors are very small. We believe this to be due to the number of data points per line being very large, resulting in a very small error for the gradient, when it was not the case. Therefore, horizontal error bars were included in the line-fitting. Using the gradient from this fit and calculating values of viscosity, the combined measured value is within two standard errors of the accepted value, which is reasonable agreement, therefore we find good reason to say that the method was valid.

5. CONCLUSIONS

The viscosity of water was determined using two different capillary tubes. Both tubes' returned values of viscosity were combined and the final value was 0.97 ± 0.05 mPa s, which was within two standard errors of the accepted value. This shows that using the Hagen-Poiseuille relation and capillary tubes is a valid method of viscometry.

If done again, the experiment would take more care in keeping the height constant and use a broader range of capillary tubes.

REFERENCES

[1] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge (1967).

[2] H. D. Young and R. A. Freedman, *University Physics with Modern Physics*, 13th Ed., Pearson Addison-Wesley, San Francisco (2012).

[3] Robert C Reid, J M Prausnitz, Bruce E Poling, *The Properties of Gases and Liquids*, 5th Ed., McGraw-Hill, New York (2001).

Appendix A: Errors Appendix

1. Measurement Uncertainties

The standard errors on measurements with a ruler was taken to be half an analogue division of the measuring device used. Whenever distances were combined, such as when the length of the capillary tubes was measured or when the depth of the water from the hole in the reservoir was measured, the errors were combined in quadrature in accordance with the equation,

$$\alpha_d = \sqrt{\alpha_{d_1}^2 + \alpha_{d_2}^2}, \tag{A1}$$

where α_{d_1} and α_{d_2} are the uncertainties on the distance measurements d_1 and d_2, respectively, and α_d is the final uncertainty on the final distance measurement d. [This equation, like all of the equations included in Appendix A, is based on the error analysis formula given in I. G. Hughes and T. P. A. Hase, *Measurements and Their Uncertainties*, Oxford University Press: Oxford (2010).]

The diameter of the capillary tubes was measured by placing a travelling microscope target on each side of the circular hole, then repeating this at different angles. These diameters were combined by calculating the mean and standard error of the individual measurements. The mean is calculated using the equation,

$$\bar{B} = \frac{1}{N}\sum_{i=1}^{N} B_i, \tag{A2}$$

where $\bar{B}$ is the mean measurement of B and B_i are individual measurements of the magnetic field density B.

The sample standard deviation, σ_{sample}, of the set of measurements is worked out using the equation,

$$\sigma_{sample} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N} d_i^2}, \tag{A3}$$

where $d_i = \bar{B} - B_i$. The uncertainty in the measurement of $\bar{B}$ is taken to be its standard error, α_B, where

$$\alpha_B = \frac{\sigma_{sample}}{\sqrt{N}}. \tag{A4}$$

2. χ^2-Minimisation for Parameter Fitting

The χ^2 statistic for a fit of y against x of y(x) is,

$$\chi^2 = \sum_i \frac{(y_i - y(x_i))^2}{\alpha_i^2}, \tag{A5}$$

where $y(x_i)$ is the measurement of y at x_i, y_i is the corresponding value from the fit (linear in this case), and α_i is the standard error on the i^{th} data point. In this experiment, this statistic is minimised using computer software to determine a best-fit for the data points. An error on this fit is determined by the extremum of the $\chi^2 + 1$ contour on a contour plot of χ^2 with the fit parameters used.

3. Combining Values - The Weighted Mean

As seen in the report, the calculated viscosities for both capillary tubes are combined using the weighted mean. In the case where two values are combined, the weighted mean x of x_1 and x_2, with errors of α_1 and α_2 respectively, is,

$$x = \frac{\alpha_1^{-2}x_1 + \alpha_2^{-2}x_2}{\alpha_1^{-2} + \alpha_2^{-2}}. \tag{A6}$$

Then, the combined error α is given by,

$$\frac{1}{\alpha^2} = \frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} \tag{A7}$$